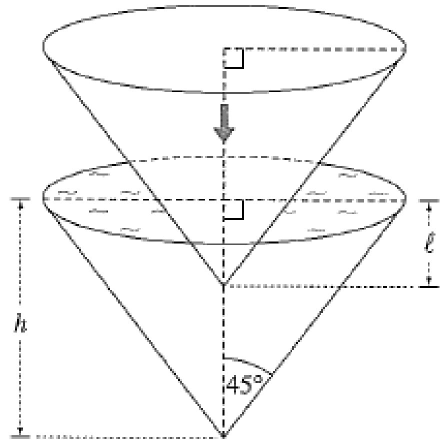
11 **7a**

The diagram shows two identical circular cones with a common vertical axis. Each cone has height h cm and semi-vertical angle 45°. The lower cone is completely filled with water. The upper cone is lowered vertically into the water as shown in the diagram. The rate at which it its lowered is given by $\frac{d\ell}{dt} = 10$, where ℓ cm is the distance the upper cone has descended into the water after t seconds. As the upper cone is lowered, water spills from the lower cone. The volume of water remaining in the lower cone at time t is V cm^3.



Not to scale

(i) Show that $V = \frac{\pi}{3}\left(h^3 - \ell^3\right)$. **1**

(ii) Find the rate at which V is changing with respect to time when $\ell = 2$. **2**

(iii) Find the rate at which V is changing with respect to time when the lower cone has lost $\frac{1}{8}$ of its water. Give your answer in terms of h. **2**

State Mean: **0.42/1** **1.01/2** **0.23/2**

(i) As semi-vertical angle is 45°, then radius = height (isos. triangle)

$\therefore$ vol. of lower cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi h^3$

$\therefore$ vol. of upper cone $= \frac{1}{3}\pi \ell^3$

$$\therefore V = \frac{1}{3}\pi h^3 - \frac{1}{3}\pi \ell^3 = \frac{\pi}{3}\left(h^3 - \ell^3\right)$$

(ii)
$$V = \frac{\pi}{3}\left(h^3 - \ell^3\right)$$
$$\frac{dV}{d\ell} = \frac{\pi}{3}\left(-3\ell^2\right) = -\pi\ell^2$$

When $\ell = 2$, $\frac{dV}{d\ell} = -4\pi$

Also, $\frac{d\ell}{dt} = 10$,

$$\frac{dV}{dt} = \frac{dV}{d\ell} \times \frac{d\ell}{dt} = -4\pi \times 10 = -40\pi$$

$\therefore -40\pi$ cm^3/sec

(iii)
$$\frac{1}{3}\pi\ell^3 = \frac{1}{8} \times \frac{1}{3}\pi h^3$$
$$\ell^3 = \frac{1}{8}h^3$$
$$\ell = \frac{h}{2}$$
$$\frac{dV}{d\ell} = -\pi\ell^2$$

When $\ell = \frac{h}{2}$, $\frac{dV}{d\ell} = -\frac{\pi h^2}{4}$. Also, $\frac{d\ell}{dt} = 10$,

$$\frac{dV}{dt} = \frac{dV}{d\ell} \times \frac{d\ell}{dt} = -\frac{\pi h^2}{4} \times 10 = -\frac{5\pi h^2}{2}$$

$\therefore -\frac{5\pi h^2}{2}$ cm^3/sec

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by the Board of Studies

Board of Studies: Notes from the Marking Centre

(i) Responses using the relationship $\tan 45° = \frac{r}{l}$ leading to $r = l$ were generally successful.

(ii) Most responses included the result $\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$ leading to the correct expression for $\frac{dV}{dt}$. In some responses the confusion between the variable l and the constant h led to an incorrect derivative.

(iii) In many responses candidates were unable to establish the relationship between the volumes involving l and h. Most unsuccessful attempts started by finding an alternate version of $\frac{dV}{dt}$, even though part (iii) followed directly from the result in part (ii).

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/