11 **7b**

The binomial theorem states that

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

(i) Show that $\displaystyle\sum_{r=1}^{n} \binom{n}{r} r x^r = nx(1+x)^{n-1}$ **2**

(ii) By differentiating the result from part (i), or otherwise, show that **2**

$$\sum_{r=1}^{n} \binom{n}{r} r^2 = n(n+1)2^{n-2}.$$

(iii) Assume now that n is even. Show that, for $n \geq 4$, **3**

$$\binom{n}{2}2^2 + \binom{n}{4}4^2 + \binom{n}{6}6^2 + \ldots + \binom{n}{n}n^2 = n(n+1)2^{n-3}$$

State Mean:
0.72/2
0.54/2
0.06/3

(i) $$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n$$

Differentiating both sides:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1}$$

Multiplying both sides by x:

$$nx(1+x)^{n-1} = \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 + \ldots + n\binom{n}{n}x^n = \sum_{r=1}^{n}\binom{n}{r} r x^r$$

(ii) $$nx(1+x)^{n-1} = \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 + \ldots + n\binom{n}{n}x^n$$

Differentiating both sides:

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \binom{n}{1} + 4\binom{n}{2}x + 9\binom{n}{3}x^2 + \ldots + n^2\binom{n}{n}x^{n-1} \quad \ldots\ldots (1)$$

Let $x = 1$: $$n.2^{n-1} + n(n-1).2^{n-2} = \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \ldots + n^2\binom{n}{n} = \sum_{r=1}^{n}\binom{n}{r}r^2 \quad \ldots.. (2)$$

$$n.2^{n-2}[2 + n - 1] = \sum_{r=1}^{n}\binom{n}{r}r^2$$

$$n.2^{n-2}[1 + n] = \sum_{r=1}^{n}\binom{n}{r}r^2$$

$$\therefore \sum_{r=1}^{n}\binom{n}{r}r^2 = n(n+1)2^{n-2}$$

(iii) Let $x = -1$ in ②:

$$n(1 + -1)^{n-1} + n(n - 1).\ -1(1 + -1)^{n-2} = \binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} - 4^2\binom{n}{4} + \ldots + n^2\binom{n}{n}$$

$$0 = \binom{n}{1} - 2^2\binom{n}{2} + 3^2\binom{n}{3} - 4^2\binom{n}{4} + \ldots + n^2\binom{n}{n} \quad \ldots\ldots ③$$

② – ③ :

$$n(n + 1)2^{n-2} = 2[2^2\binom{n}{2} + 4^2\binom{n}{4} + \ldots + \binom{n}{n}n^2]$$

$$n(n + 1)2^{n-3} = 2^2\binom{n}{2} + 4^2\binom{n}{4} + \ldots + \binom{n}{n}n^2$$

$$\binom{n}{2}2^2 + \binom{n}{4}4^2 + \binom{n}{6}6^2 + \ldots + \binom{n}{n}n^2 = n(n + 1)2^{n-3}$$

Board of Studies: Notes from the Marking Centre

(i) Better responses included expressions such as 'find the derivative' and 'multiply by x'. Responses in which candidates multiplied by x first and then differentiated were largely unsuccessful.

(ii) Most candidates were able to differentiate $\sum_{r=1}^{n}\binom{n}{r}rx^r$ correctly, although careless errors were common. Most candidates who attempted this part were able to recognise that the next step involved substituting $x = 1$.

(iii) Responses in which candidates attempted to prove the result using mathematical induction were generally unsuccessful.

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/