13 **14 b**

(i) Write down the coefficient of x^{2n} in the binomial expansion of $(1 + x)^{4n}$. **1**

(ii) Show that $(1 + x^2 + 2x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x + 2)^{2n-k}$. **2**

(iii) It is known that **3**

$$x^{2n-k} (x + 2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k}\, x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1}\, x^{2n-k-1}$$

$$+ \ldots + \binom{2n-k}{2n-k} 2^0\, x^{4n-2k} \text{ (Do NOT prove this.)}$$

Show that $\binom{4n}{2n} = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k} \binom{2n-k}{k}$.

(i) Coefficient of x^{2n} in $(1 + x)^{4n} = \binom{4n}{2n}$

(ii) $(1 + x^2 + 2x)^{2n} = (1 + (x^2 + 2x))^{2n}$ which is of form $(1 + X)^{2n}$ with $X = x^2 + 2x$

$$= \sum_{k=0}^{2n} \binom{2n}{k} 1^k (x^2 + 2x)^{2n-k}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x + 2)^{2n-k}$$

State Mean:
0.72/1
0.60/2
0.37/3

(iii) Firstly, $(1 + x^2 + 2x)^{2n} = (x^2 + 2x + 1)^{2n}$

$$= ((1 + x)^2)^{2n}$$

$$= (1 + x)^{4n}$$

$\therefore \binom{4n}{2n}$ is coefficient of x^{2n} in $(1 + x)^{4n}$

$$(1 + x^2 + 2x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x + 2)^{2n-k}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} \left[\binom{2n-k}{0} 2^{2n-k}\, x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1}\, x^{2n-k+1} + \ldots + \binom{2n-k}{2n-k} 2^0\, x^{4n-2k} \right]$$

Need to find all coefficients of x^{2n}, by letting $k = 0, 1, 2, \ldots, n$:

As $k = 0$, then the term involving x^{2n} is $\binom{2n}{0}\binom{2n}{0} 2^{2n}\, x^{2n}$

As $k = 1$, then the term involving x^{2n} is $\binom{2n}{1}\binom{2n-1}{1} 2^{2n-2}\, x^{2n}$

and so on …

As $k = n$, then the term involving x^{2n} is $\binom{2n}{n}\binom{2n-n}{n} 2^0\, x^{2n}$

Collecting all coefficients of x^{2n}: $\sum_{k=0}^{2n} \binom{2n}{k} \left[\binom{2n}{0}\binom{2n}{0} 2^{2n} + \binom{2n}{1}\binom{2n-1}{1} 2^{2n-2} + \ldots + \binom{2n}{n}\binom{2n-n}{n} 2^0 \right]$

$$\therefore \binom{4n}{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \left[\binom{2n}{0}\binom{2n}{0} 2^{2n} + \binom{2n}{1}\binom{2n-1}{1} 2^{2n-2} + \ldots + \binom{2n}{n}\binom{2n-n}{n} 2^0 \right] = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k}\binom{2n-k}{k}$$

Board of Studies: Notes from the Marking Centre

(i) Most candidates successfully used the binomial theorem to find the correct coefficient.

(ii) Most candidates could see that they needed to rearrange $(1+x^2+2x)^{2n}$ to $[1+x(x+2)]^{2n}$ in order to get the desired result.

The most common mistake was to use the binomial theorem to obtain

$$\sum_{k=0}^{2n}\binom{2n}{k}(1)^{2n-k}[x(x+2)]^k \text{ (or similar)}$$

and then to simply state that this was equal to $\sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}(x+2)^{2n-k}$,

with no explanation as to how the index changed from k to $2n-k$.

(iii) Very few candidates obtained full marks for this part.

Many candidates gained some marks by recognising the connection between parts (i), (ii) and (iii) and realising that $(1+x^2+2x)^{2n} \equiv (1+x)^{4n}$.

Substituting the given result into the right-hand side of the expression created problems, with incorrect statements such as $\sum_{k=0}^{2n}\binom{2n}{k}\sum_{r=0}^{2n-k}\binom{2n-k}{r}2^{2n-k-r}x^{2n-k+r}$

or other expressions involving double sums being used.

Source: http://www.boardofstudies.nsw.edu.au/hsc_exams/