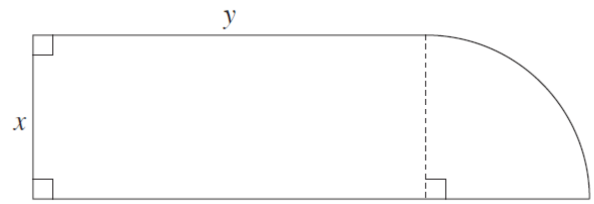
20 MA **25** A landscape gardener wants to build a garden bed in the shape of a rectangle attached to a quarter-circle.

Let x and y be the dimensions of the rectangle in metres, as shown in the diagram.

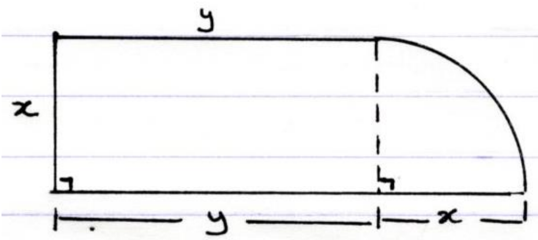


The garden bed is required to have an area of 36 m^2 and to have a perimeter which is as small as possible. Let P metres be the perimeter of the garden bed.

(a) Show that $P = 2x + \dfrac{72}{x}$. **3**

(b) Find the smallest possible perimeter of the garden bed, showing why this is the minimum perimeter. **4**

(a)



$A = xy + \frac{1}{4} \times \pi \times x^2 = 36$ ✓

$xy = 36 - \dfrac{\pi x^2}{4}$

$y = \dfrac{36}{x} - \dfrac{\pi x}{4}$ ✓

$P = 2x + 2y + \frac{1}{4} \times 2 \times \pi \times x$

$= 2y + 2x + \dfrac{\pi x}{2}$

Substitute $y = \dfrac{36}{x} - \dfrac{\pi x}{4}$:

$P = 2(\dfrac{36}{x} - \dfrac{\pi x}{4}) + 2x + \dfrac{\pi x}{2}$

$= \dfrac{72}{x} - \dfrac{\pi x}{2} + 2x + \dfrac{\pi x}{2}$

$= 2x + \dfrac{72}{x}$ ✓

(b) $P = 2x + \dfrac{72}{x}$

$= 2x + 72x^{-1}$

$\dfrac{dP}{dx} = 2 - 72x^{-2}$ ✓

$= 2 - \dfrac{72}{x^2} = 0$

$2 = \dfrac{72}{x^2}$

$2x^2 = 72$

$x = 6 \ (x > 0)$ ✓

$\dfrac{d^2P}{dx^2} = 144x^{-3}$

$= \dfrac{144}{x^3}$

$\dfrac{d^2P}{dx^2}(6) = \dfrac{144}{6^3} > 0.$

Hence, minimum perimeter when $x = 6$. ✓

$P(6) = 2(6) + \dfrac{72}{6}$

$= 24$

$\therefore$ the smallest possible perimeter is 24 m. ✓

State Mean:
1.40/3
2.20/4

HSC Marking Feedback

Question 25 (a)

Students should:

- use algebraic techniques to construct expressions representing area and perimeter of familiar shapes
- rearrange equations involving fractions to change the subject
- substitute the expression for y in terms of x into P
- simplify algebraically to achieve the desired result.

In better responses, students were able to:

- define variables and develop functions representing the area and perimeter of the composite shape in terms of x and y
- make y the subject and simplify this expression before substituting into P
- simplify the expression for y before substituting into P
- substitute and simplify the algebraic expressions effectively
- show all working.

Areas for students to improve include:

- finding the arc length in terms of x
- simplifying, collecting like terms, and taking a common denominator when manipulating complex algebraic equations
- deriving the area and perimeter equations in terms of π for a quadrant
- solving simultaneously to find P in terms of x
- acknowledging the 'show that' as a checking mechanism rather than forcing equations to achieve the desired result.

Question 25 (b)

Students should:

- solve optimisation problems involving perimeter
- use calculus to find the derivative
- solve the equation after equating the derivative to zero
- verify the minimum value of x by using either the first or second derivative
- substitute the value of x obtained from calculus to find the minimum perimeter
- provide reasoning to support conclusions in the given context.

In better responses, students were able to:

- rearrange an expression using index laws before differentiating
- differentiate correctly to solve $P' = 0$
- use the first derivative table using numeric values for P' to verify minimum perimeter
- calculate a positive second derivative after substituting the value of x to determine concavity verifying minimum perimeter
- answer the question by calculating the minimum P
- minimise using calculus despite being unsuccessful in part (a).

Areas for students to improve include:

- differentiating fractions involving x in the denominator
- checking calculations when the side length is negative
- explicitly including correct notation of P' and calculating numeric values in the first derivative table
- formulating relevant conclusions by stating that P is minimised when $P''(x) > 0$
- answering the question to find the minimum perimeter.

* These solutions have been provided by *projectmaths* and are not supplied or endorsed by NESA.